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EC 4214

Wave Equation : (Maxwell equation)

$$\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad (1)$$

Sol) $E(z,t) = E_0(z,t)\, e^{i(kz - \omega t + \phi(z,t))}$,

where E_0 and ϕ are slowly varying.

(i) ∇E : $\frac{\partial E}{\partial z} = \left[\frac{\partial E_0}{\partial z} + E_0\left(ik + i\frac{\partial \phi}{\partial z}\right)\right] e^{i(kz - \omega t + \phi(z,t))}$

$\nabla^2 E$: $\frac{\partial^2 E}{\partial z^2} = \left[\cancel{\frac{\partial^2 E_0}{\partial z^2}}^{\,0} + i\frac{\partial E_0}{\partial z}\left(k + \cancel{\frac{\partial \phi}{\partial z}}^{\,0}\right) + iE_0\cancel{\frac{\partial^2 \phi}{\partial z^2}}^{\,0}\right] e^{i(kz - \omega t + \phi(z,t))}$

$+ \left[\frac{\partial E_0}{\partial z} + iE_0\left(k + \frac{\partial \phi}{\partial z}\right)\right]\left(i\left(k + \frac{\partial \phi}{\partial z}\right)\right) e^{i(kz - \omega t + \phi(z,t))}$

$= \left[ik\frac{\partial E_0}{\partial z} + ik\frac{\partial E_0}{\partial z} - k^2 E_0 + ikE_0\frac{\partial \phi}{\partial z} + ik\frac{\partial \phi}{\partial z}E_0\right] e^{i(kz - \omega t + \phi(z,t))}$

$= \left(2ik\frac{\partial E_0}{\partial z} + 2ikE_0\frac{\partial \phi}{\partial z} - k^2 E_0\right) e^{i(kz - \omega t + \phi(z,t))}$

Here, second order derivatives go to zero due to $\begin{cases} \frac{\partial E_0}{\partial z} \ll kE_0 \,;\; \frac{\partial E}{\partial t} \ll \omega E \\ \frac{\partial \phi}{\partial z} \ll k \,;\; \frac{\partial \phi}{\partial t} \ll \omega \end{cases}$

(ii) $\frac{\partial E}{\partial t} = \left[\frac{\partial E_0}{\partial t} + iE_0\left(-\omega + \frac{\partial \phi}{\partial t}\right)\right] e^{i(kz - \omega t + \phi(z,t))}$

$\frac{\partial^2 E}{\partial t^2} = \left(\cancel{\frac{\partial^2 E_0}{\partial t^2}}^{\,0} + i\frac{\partial E_0}{\partial t}\left(-\omega + \cancel{\frac{\partial \phi}{\partial t}}^{\,0}\right) + iE_0\cancel{\frac{\partial^2 \phi}{\partial t^2}}^{\,0}\right) e^{i(kz - \omega t + \phi(z,t))}$

$+ \left[\frac{\partial E_0}{\partial t} + iE_0\left(-\omega + \frac{\partial \phi}{\partial t}\right)\right]\left[i\left(-\omega + \frac{\partial \phi}{\partial t}\right)\right] e^{i(kz - \omega t + \phi(z,t))}$

$= \left[-i\omega\frac{\partial E_0}{\partial t} - i\omega\frac{\partial E_0}{\partial t} - \omega^2 E_0 - i\omega E_0\frac{\partial \phi}{\partial t} - i\omega E_0\frac{\partial \phi}{\partial t}\right] e^{i(kz - \omega t + \phi(z,t))}$

$= \left(-2i\omega\frac{\partial E_0}{\partial t} - 2i\omega E_0\frac{\partial \phi}{\partial t} - \omega^2 E_0\right) e^{i(kz - \omega t + \phi(z,t))}$

By inserting (i) & (ii) into the wave equation (1) :

$$2ik\frac{\partial E_0}{\partial z} + 2ikE_0\frac{\partial \phi}{\partial z} - k^2 E_0 = \frac{-1}{c^2}\left(+2i\omega\frac{\partial E_0}{\partial t} + 2i\omega E_0\frac{\partial \phi}{\partial t} + \omega^2 E_0\right)$$

The only solution is as follows :

$$\begin{cases} 2ik\frac{\partial E_0}{\partial z} + \frac{1}{c^2}(2i\omega)\frac{\partial E_0}{\partial t} = (k^2 - \omega^2/c^2)E_0 = 0 \,; \quad c = \frac{\omega}{k} \\ 2ik\frac{\partial \phi}{\partial z} + \frac{1}{c^2}\left(2i\omega\frac{\partial \phi}{\partial t}\right) = 0 \end{cases}$$

$\therefore$ Maxwell-Bloch equations are:

$$\frac{\partial E}{\partial z} + \frac{1}{c}\frac{\partial E_0}{\partial t} = 0 \;;\quad \frac{\partial \phi}{\partial z} + \frac{1}{c}\frac{\partial \phi}{\partial t} = 0$$