〈 물질파 vs. Schrödinger's wave eq. 〉 2021. 3. 15

(i) From Einstein's speciality relativity theory, (1905)

$E = hf$ & $E = pc$

From Maxwell's eq.

$\lambda f = c$

Combining them for E,

$hf = pc = p\lambda f \rightarrow \boxed{\lambda = h/p}$ de Broglie's wave-particle duality (1924)

(ii) $\nabla^2 E - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$

Sol. $E(x,t) = E_0 e^{i(kx-\omega t)}$: Maxwell eq

$\nabla^2 E = -ik^2 E$; $\frac{\partial^2 E}{\partial t^2} = -\omega^2 E$

$\rightarrow k^2 = \omega^2/c^2$ $\therefore \boxed{\omega = ck}$: dispersion relation

(iii) $E_0 e^{i(kx-\omega t)} \rightarrow E_0 e^{+\frac{i}{\hbar}(px-Et)}$; $p = \hbar k$; $E = \hbar\omega$

$\nabla^2 E = -\frac{p^2}{\hbar^2}$; $\frac{\partial^2 E}{\partial t^2} = -\frac{E^2}{\hbar^2}$

$\rightarrow p^2 = \frac{E^2}{c^2}$ $\therefore E = pc$ ok!

(energy-momentum relation)

(iv) Wave eq. for a particle

$\varphi(x,t) = Ae^{\frac{i}{\hbar}(px-Et)}$;

Using $E^2 = p^2c^2 + m_0^2c^4$ (relativistic theory)

$\rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m_0^2c^2}{\hbar^2}\right)Ae^{\frac{i}{\hbar}(px-Et)} = -\frac{1}{\hbar^2}\left(p^2 - \frac{E^2}{c^2} + m_0^2c^2\right)\varphi(x,t)$

$= 0$: Klein-Gordon eq.

(v) Non-relativistic wave eq.

$\psi(x,t) = Ae^{\frac{i}{\hbar}(px-Et)}$

$\frac{\partial \psi}{\partial t} = -\frac{iE}{\hbar}\psi$; $\frac{\partial^2 \psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi$

Using $E = \frac{p^2}{2m}$, $\frac{\partial^2 \psi}{\partial x^2} = -\frac{2mE}{\hbar^2}\psi$

$\rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} = E\psi = -\frac{\hbar}{i}\frac{\partial \psi}{\partial t} = i\hbar\dot{\psi}$ $\therefore i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\nabla^2\psi$

(Schrödinger's Eq.)