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11-4. Resolution

· Airy disk : 1st maximum image through a circular aperature.
→ resolution limit of an image.

· Rayleigh's criterion : Max of the 2nd falls on the 1st min of the other.

$$\Delta\theta_{min} = \Delta\theta_{1/2}\ (\text{Airy disk}) = \frac{1.22}{D}\lambda$$

ex) lens diameter : 35 mm in binoculars
What is min seperation of two stars to be resolvable?

Sol)
$$(\Delta\theta)_{min} = \frac{1.22\lambda}{D} = \frac{(1.22)(550\times10^{-9})}{35\times10^{-3}} = 1.92\times10^{-5}\ (\text{rad})$$

ex) Microscope
— objective lens focal length : f
What is min seperation of two objects to be resolvable?

Sol)
$$(\Delta\theta)_{min} = \frac{1.22\lambda}{D}$$

$$\therefore\ x_{min} = f(\Delta\theta)_{min} = f\left(\frac{1.22\lambda}{D}\right)$$

→ $\frac{D}{f}$: numerical aper·ture (~1.2)

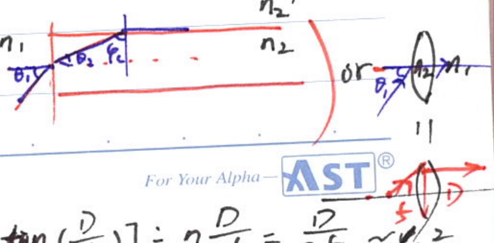
→ N.A. : For maximum collection angle of incidence light

$\underline{N.A. \equiv n_1 \sin\theta_1} = n_2 \sin\theta_2$, $\theta_2 = \frac{\pi}{2} - \varphi_c$; φ_c : critical angle

$= n_2 \cos\varphi_c$

· $n_2 \sin\varphi_c = n_2' \sin 90 \rightarrow \sin\varphi_c = \left(\frac{n_2'}{n_2}\right)$

$\therefore \cos\varphi_c = \frac{\sqrt{n_2^2 - n_1'^2}}{n_2}$, $n_1' = n_2'$



$$\therefore\ N.A. = \sqrt{n_2^2 - n_1^2} = n_1 \sin\theta_1 = n_1 \sin\left[\arctan\left(\frac{D}{2f}\right)\right] \doteq n\frac{D}{2f} = \frac{D}{2f} \sim 1.2$$

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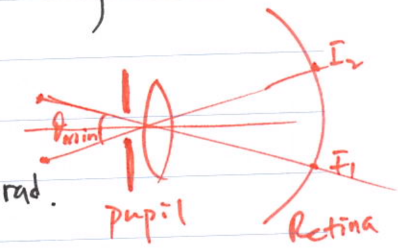
$\therefore \; \underline{\underline{x_{min} \sim \lambda}}$. (resolution of microscopy)

ex) Night vision vs Day vision

- pupil's diameter : 2 mm in a bright day
  8 mm at night



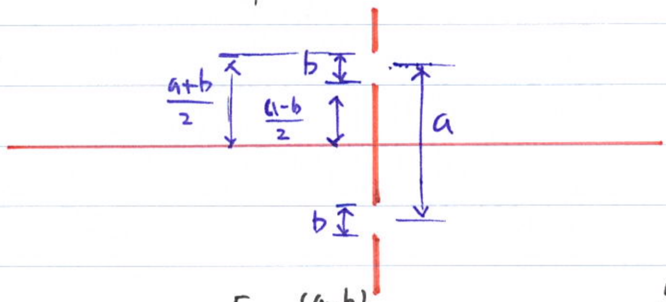
Day (2mm D)

$\rightarrow (\Delta\theta)_{min} = \dfrac{1.22\lambda}{D} = \dfrac{(1.22)(550\times10^{-9})}{2\times10^{-3}} = 0.34$ mrad.

$\rightarrow$ Objects at 2m away. $\equiv l$

$\Delta\theta_{min} = \dfrac{\Delta y}{l} = \dfrac{\Delta y}{2}$, $\therefore \; \Delta y = 2(\Delta\theta_{min}) = 0.68\,(mm)$

11-5 Double slit exp (Diffraction)



$$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \left[\int_{-\frac{(a+b)}{2}}^{-\frac{(a-b)}{2}} e^{isk\sin\theta} ds + \int_{\frac{a-b}{2}}^{\frac{a+b}{2}} e^{isk\sin\theta} ds \right]$$

$$= \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \frac{1}{ik\sin\theta} \left[e^{ik\sin\theta\left(\frac{-a+b}{2}\right)} - e^{ik\sin\theta\left(\frac{-a-b}{2}\right)} + e^{ik\sin\theta\left(\frac{a+b}{2}\right)} - e^{ik\sin\theta\left(\frac{a-b}{2}\right)} \right]$$

let $\beta \equiv \frac{1}{2} kb\sin\theta$; $\alpha \equiv \frac{1}{2} ka\sin\theta$

$$= \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \frac{b}{2i\beta} \left[e^{-i\alpha}(e^{i\beta} - e^{-i\beta}) + e^{i\alpha}(e^{i\beta} - e^{-i\beta}) \right]$$

$$= \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \frac{b}{2i\beta} \left[i2\sin\beta \, (e^{i\alpha} + e^{-i\alpha}) \right]$$

$$= \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \frac{b}{2i\beta} (2i\sin\beta)(2\cos\alpha)$$

$$= \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \left(\frac{\sin\beta}{\beta} \right) (2b)(\cos\alpha)$$

$$\therefore I = \left(\frac{\epsilon_0 c}{2} \right) E_p E_p^* = \left(\frac{\epsilon_0 c}{2} \right) \left(\frac{2E_L b}{r_0} \right)^2 \operatorname{sinc}^2\beta (\cos\alpha)^2$$

$$= 4I_0 \operatorname{sinc}^2\beta \cos^2\alpha$$

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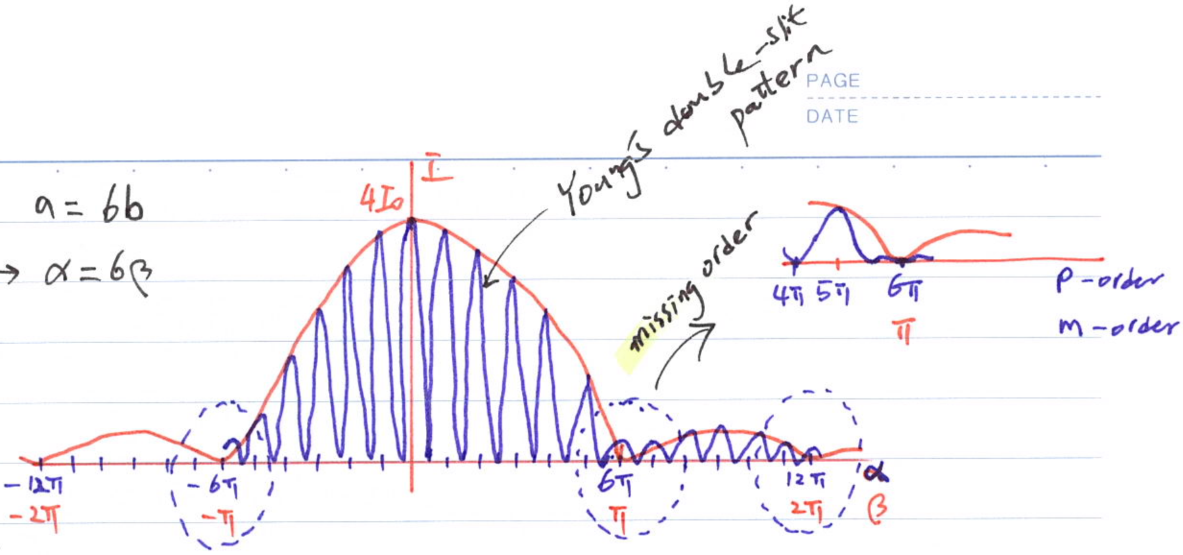
For $a = 6b$

$\rightarrow \alpha = 6\beta$



For zeros,

(i) for α, $\alpha = \frac{1}{2} k a \sin\theta$

$\cos^2\alpha = 0 \rightarrow \alpha = (m + \frac{1}{2})\pi$, $m = 0, \pm 1, \pm 2, \cdots$

(ii) for β, $\beta = \frac{1}{2} k b \sin\theta$

$\text{sinc}^2\beta = 0 \rightarrow \beta = m\pi$, $m = \pm 1, \pm 2, \cdots$

$\rightarrow \cos^2\alpha = \cos^2\left(\frac{k a \sin\theta}{2}\right) = \cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$

$\rightarrow$ See 7-21,

$I = 4I_0 \cos^2\left(\frac{\pi a \sin\theta}{\lambda}\right)$

(Young's double-slit)

· Diffraction minima for β,

$\beta = \frac{1}{2} k b \sin\theta = \frac{\pi b \sin\theta}{\lambda} = m\pi \rightarrow m\lambda = b\sin\theta$.

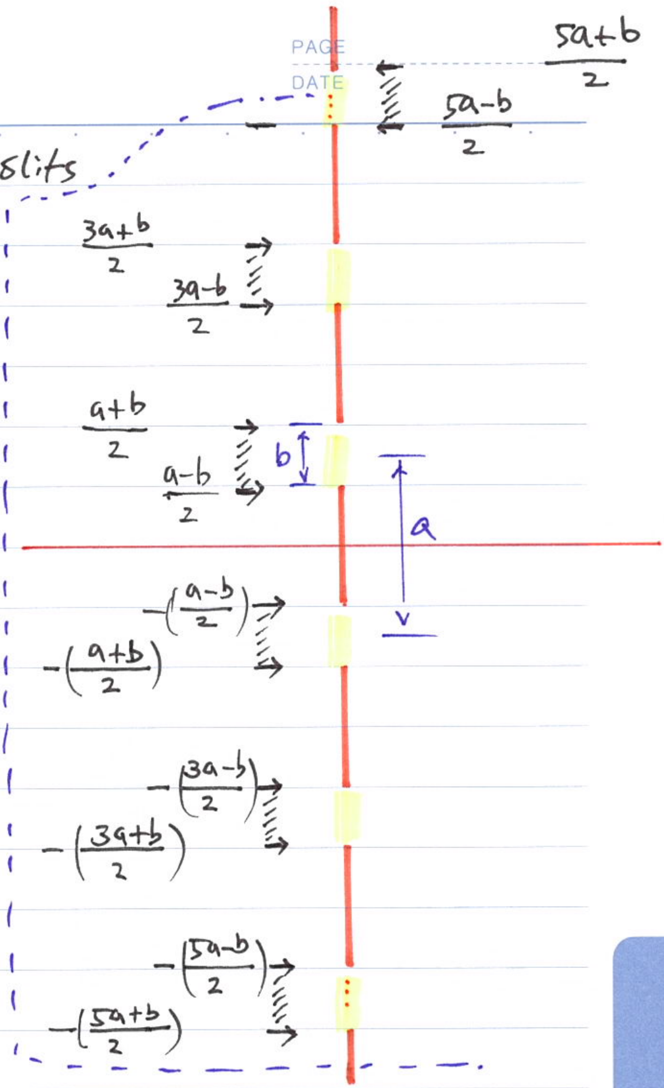
· Interference maxima for α,

$\alpha = \frac{1}{2} k a \sin\theta = \frac{\pi a \sin\theta}{\lambda} = p\pi \rightarrow p\lambda = a\sin\theta$

☆ Missing order : $\frac{m}{b} = \frac{p}{a} \rightarrow a = \left(\frac{p}{m}\right) b$

(max → min). ex) If $a = b$, $p = m$ ∴ all orders are missing except $p=0$.

11-6 Diffraction from many slits



- $$E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \sum_{j=1}^{N/2} \{A+B\}$$

- $$A = \int_{-\left[\frac{(2j-1)a-b}{2}\right]}^{-\left[\frac{(2j-1)a+b}{2}\right]} e^{isk\sin\theta}\, ds$$

- $$B = \int_{\frac{(2j-1)a-b}{2}}^{\frac{(2j-1)a+b}{2}} e^{isk\sin\theta}\, ds$$

ex)

$j=1 \rightarrow$ double slit

$j=2 \rightarrow$ quad slit

- $$A = \frac{1}{ik\sin\theta}\left\{ e^{-ik\sin\theta\left[\frac{(2j-1)a-b}{2}\right]} - e^{-ik\sin\theta\left[\frac{(2j-1)a+b}{2}\right]} \right\}$$

- $$B = \frac{1}{ik\sin\theta}\left\{ e^{ik\sin\theta\left[\frac{(2j-1)a+b}{2}\right]} - e^{+ik\sin\theta\left[\frac{(2j-1)a-b}{2}\right]} \right\}$$

$\alpha = \frac{1}{2}ka\sin\theta$; $\beta = \frac{1}{2}kb\sin\theta$

$$\therefore A+B = \frac{b}{2i\beta}\left[e^{-i(2j-1)\alpha}\left(e^{i\beta} - e^{-i\beta}\right) + e^{i(2j-1)\alpha}\left(e^{i\beta} - e^{-i\beta}\right)\right]$$

$$= \frac{b}{\cancel{2i}\beta}\left\{ \cancel{2i}\sin\beta \cdot \underline{2\cos[(2j-1)\alpha]} \right\}$$

$$= 2b\,\mathrm{sinc}(\beta)\,\mathrm{Re}\left(e^{i(2j-1)\alpha}\right)$$

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$\cdot \quad E_p = \frac{E_L}{r_0} e^{i(kr_0 - \omega t)} \; 2b\,\mathrm{sinc}(\beta)\; \mathrm{Re} \underline{\underline{\sum_{j=1}^{N/2} e^{i(2j-1)\alpha}}}$

$\downarrow$

$$\sum_{j=1}^{N/2} e^{i(2j-1)\alpha} = e^{i\alpha} + e^{i3\alpha} + e^{i5\alpha} + \cdots + e^{i(N-1)\alpha}$$

$$= e^{i\alpha}\,\underline{\underline{(1 + e^{i2\alpha} + e^{i4\alpha} + \cdots)}}$$

cf. $\sum_{j=0}^{N/2} x^j = \frac{1-x^{N/2}}{1-x}$

Let $x = e^{2i\alpha}$

$$\rightarrow \frac{1-e^{iN\alpha}}{1-e^{i2\alpha}}$$

$\underline{* \; e^{i\alpha}}$

$$\rightarrow e^{i\alpha}\left(\frac{1-e^{iN\alpha}}{1-e^{i2\alpha}}\right) = \frac{1-e^{iN\alpha}}{e^{-i\alpha}-e^{i\alpha}}$$

$$= \frac{1-\cos N\alpha - i\sin N\alpha}{-2i\sin\alpha} \qquad \times(\;)\left(\frac{-i}{-i}\right)$$

$$= \frac{i(\cos N\alpha - 1) - \sin N\alpha}{-2\sin\alpha}$$

$\therefore$ Real part : $\frac{\sin N\alpha}{2\sin\alpha}$.

$$\therefore E_p = \frac{E_L}{r_0} e^{i(kr_0-\omega t)}\; b\,\mathrm{sinc}(\beta)\left(\frac{\sin N\alpha}{\sin\alpha}\right)$$

$$\therefore I = I_0 \underbrace{\mathrm{sinc}^2(\beta)}_{\text{Diffraction}} \underbrace{\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2}_{\text{Interference}}$$

$N=1$: Single slit
$N=2$: Double slit

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

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· $\alpha \equiv \frac{1}{2} k a \sin\theta$

for $\lim_{x\to c} f(x) = \lim_{x \to c} g(x) = 0$,

· L'Hopital's rule : $\lim_{x\to c}\left(\frac{g(x)}{f(x)}\right) = \lim_{x\to c}\left(\frac{g'(x)}{f'(x)}\right)$.

principal max: $\lim_{\alpha\to m\pi} \frac{\sin N\alpha}{\sin\alpha} = \lim_{\alpha \to m\pi} \frac{N\cos N\alpha}{\cos\alpha} = N$

$\rightarrow$ Condition for $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$: $c = \pm m\pi$

$\therefore I = I_0\left(\frac{\sin\beta}{\beta}\right)^2\left(\frac{\sin N\alpha}{\sin\alpha}\right)^2$

At $\alpha \to \pm m\pi$, $I = I_0 \,\mathrm{sinc}^2(\beta)\, \underline{N^2}$

ex) $N = 8$, & $a = 3b$

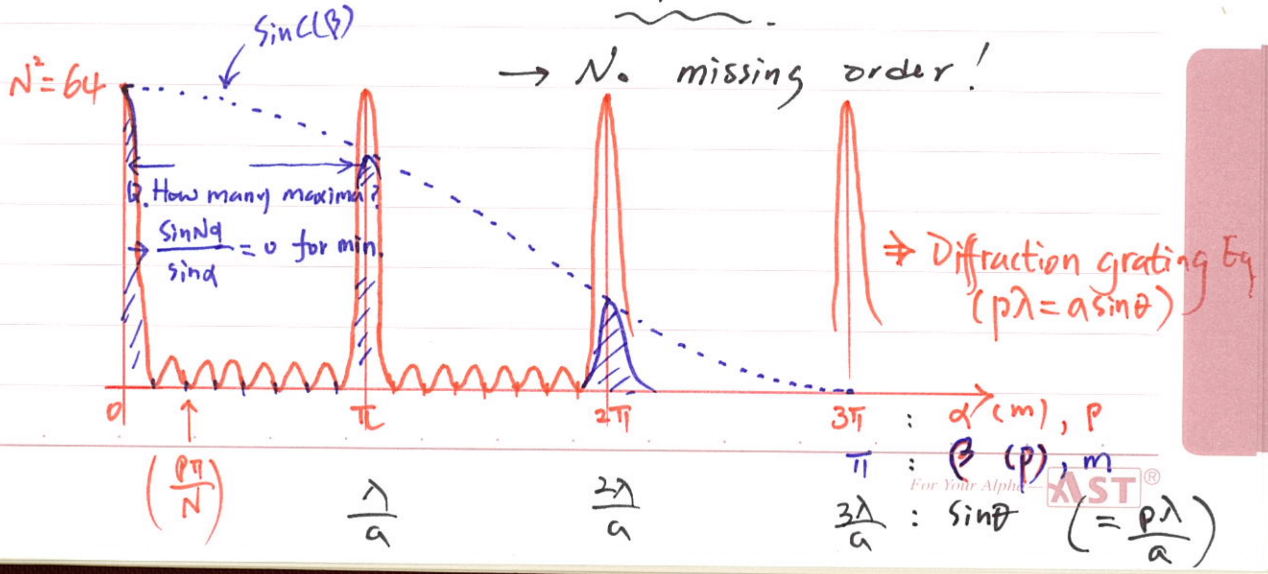
· $\beta \equiv \frac{1}{2} k b \sin\theta = \frac{\pi}{\lambda} b\sin\theta \rightarrow b\sin\theta = m\lambda$ (max)

· $\alpha = \frac{1}{2} k a \sin\theta = \frac{\pi}{\lambda} a \sin\theta \rightarrow \underline{a\sin\theta = p\lambda}$ (max)

$\rightarrow \frac{b}{m} = \frac{a}{p}$, $\rightarrow a = \left(\frac{p}{m}\right) b$

$\therefore \frac{p}{m} = 3$ or $p = 3m$

$\rightarrow$ No missing order!



< The secondary maxima calculation. btwn two principal maxima >

· For zeros,

From $\frac{\sin N\alpha}{\sin\alpha}$,

$\sin N\alpha = 0$ but $\sin\alpha \neq 0$.

$\rightarrow N\alpha = p\pi \rightarrow \alpha = p\pi/N$; $p = 1, 2, 3, \cdots$

For $N = 8$,

$\alpha = \frac{0}{8}\pi, \frac{1}{8}\pi, \frac{2}{8}\pi, \frac{3}{8}\pi, \frac{4}{8}\pi, \frac{5}{8}\pi, \frac{6}{8}\pi, \frac{7}{8}\pi$ and $\frac{8}{8}\pi$

except $\alpha = \frac{0}{8}\pi$ & $\frac{8}{8}\pi$, all αs result in zero of $\sin N\alpha$,

$\rightarrow$ # of minima (secondary) : $N - 2 = \underline{\underline{6}}$.