Ch. 4. Wave Equations

- Mathematical expression of wave motion (traveling waves)
- Harmonic waves
- Energy delivery via waves

4-1. One dimensional wave Eq.

· For a traveling wave, whose velocity is v,
  the notation of v is with respect to the phase or
  a fixed point in a ~~fixed~~ moving frame of O'.
  → y value must not be changed for the fixed point
  if O' moves along $\hat{x}$.

Thus,

$$y = y' = f(x') = f(x - vt) \qquad : \underline{x' = x - vt}.$$

· Math notation

$$\# \quad \frac{\partial y}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} = \frac{\partial f}{\partial x'}, \quad \text{where } \underline{\frac{\partial x'}{\partial x} = 1}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \frac{\partial}{\partial x'}\left(\frac{\partial x'}{\partial x}\right)\left[\frac{\partial f}{\partial x'}\right] = \frac{\partial^2 f}{\partial x'^2} \qquad ①$$

$$\# \quad \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t} = -v\frac{\partial f}{\partial x'}, \quad \text{where } \frac{\partial x'}{\partial t} = -v$$

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x'}\left(\frac{\partial x'}{\partial t}\right)\left[-v\frac{\partial f}{\partial x'}\right] = v^2\frac{\partial^2 f}{\partial x'^2} \qquad ②$$

$$\therefore \boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} \qquad \text{(wave equation)} \ ☆$$

4-2. Harmonic waves

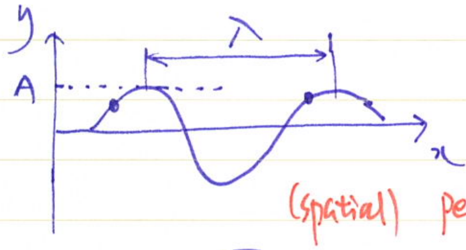
- Let a traveling wave be:

$$y = A\sin[k(x \pm vt)]$$

: A, amplitude
  k, wave number (vector)
  v, moving velocity

- Harmonic means peoridic.
  ex) undamped oscillators
- Superposition of harmonic waves are also harmonic.
  → Fourier series (in CH.9)

A.



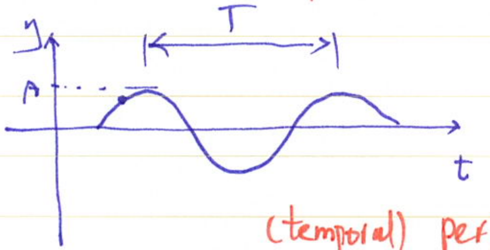
: t-fixed, λ: wavelength

(spatial) periodicity: $A\sin k[(x+\lambda)+vt]$

$= A\sin[k(x+vt)+2\pi]$

$\to \boxed{\left(k = \frac{2\pi}{\lambda}\right)}$

B.



: x-fixed, T: period

(temporal) periodicity: $A\sin k[x+v(t+T)]$

$= A\sin[k(x+vt)+2\pi]$

$\to kvT = 2\pi$

$\to \left(\frac{2\pi}{\lambda}\right)(f\lambda)T = 2\pi$

$\to f = \frac{1}{T} = \nu$ (nu)

$\kappa = \frac{1}{\lambda}$: wave number

$\nu = \frac{1}{T}$: frequency

→ $\omega = 2\pi\nu$: angular freq.

Thus the original traveling wave can be rewritten as :

$$y = A\sin[k(x \pm vt)]$$

$$= A\sin\left[\frac{2\pi}{\lambda}(x \pm \lambda f t)\right]$$

$$= A\sin\left[\frac{2\pi}{\lambda}(x) \pm 2\pi f t\right] = A\sin(kx \pm \omega t)$$

$\rightarrow$ $\varphi = k(x \pm vt) = kx \pm \omega t$: phase

* Constant phase describes velocity of the wave.

$\rightarrow$ $d\varphi = 0$ $\rightarrow$ $k(dx \pm v\,dt) = 0$

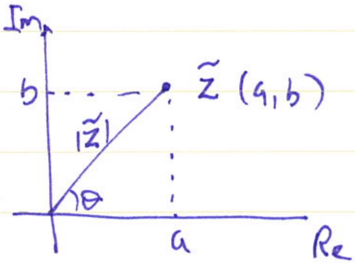
$\therefore$ $\dfrac{dx}{dt} = \mp v$ ($+v \rightarrow$ along $+\hat{x}$)

(positive direction : $A\sin(kx - \omega t)$
negative " : $A\sin(kx + \omega t)$.

4-3. Complex numbers

· $\tilde{z}$: complex number

$\tilde{z} = a + ib$, $i = \sqrt{-1}$: Euler's formula



$|\tilde{z}|^2 = a^2 + b^2$, $a = |\tilde{z}|\cos\theta$

$b = |\tilde{z}|\sin\theta$

$\therefore$ $\tilde{z} = |\tilde{z}|(\cos\theta + i\sin\theta) = |\tilde{z}|e^{i\theta}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

· Complex conjugate $\tilde{z}^*$

$\tilde{z}^* = a - ib$; $\tilde{z}^* = |\tilde{z}|e^{-i\theta}$; $\tilde{z}\tilde{z}^* = |\tilde{z}|^2$

4-4. Harmonic waves as complex fn

$$\tilde{y} = A e^{i(kx - \omega t)}$$

$\rightarrow$ $Re(\tilde{y}) = A\cos(kx - \omega t)$; $Im(\tilde{y}) = A\sin(kx - \omega t)$

4-5. plane waves

- In a 3-D space,

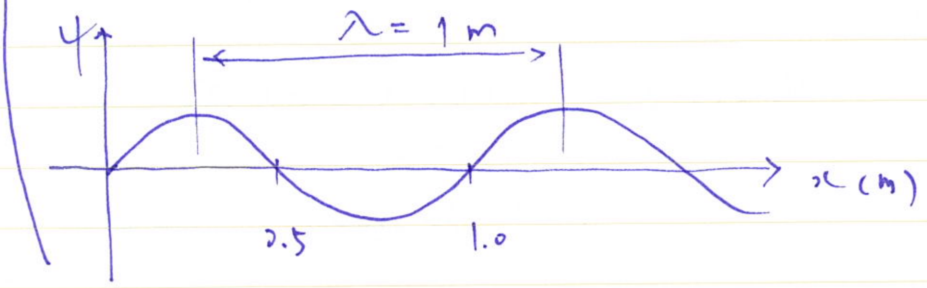
$$\psi(x, y, z) = A\sin(kr\cos\theta) \quad \text{for } t = 0$$

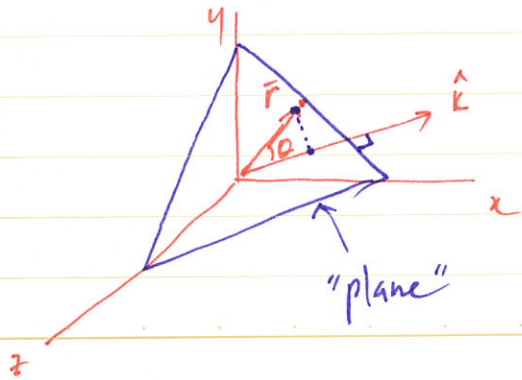
$$\text{or } = A\sin(\bar{k}\cdot\bar{r}) \quad ; \quad \bar{k}\cdot\bar{r} = k_x x + k_y y + k_z z$$

$\rightarrow$ In a 1-D (x),

$\psi = A\sin kx$ $\longrightarrow$ $\varphi = kx = \text{const.}$ if $x = \text{const.}$

"displacement" $\longrightarrow$ "plane" waves (or wavefront)
(transverse to $\hat{k}$)





$$\psi(\bar{r}, t) = A e^{i(\bar{k}\cdot\bar{r} - \omega t)}$$

$$\rightarrow \nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$$

4-6. Spherical waves

- Originated in a point source for a homogeneous & isotropic medium

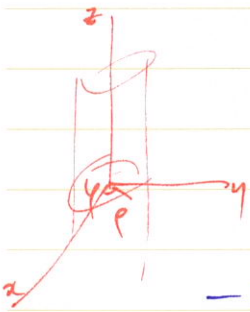
$$\psi = \left(\frac{A}{r}\right) e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$; r : radial distance from the point source

→ power decreases as $\left(\frac{1}{r}\right)^2$: inverse square law

→ $r \gg \lambda$, considered as a plane wave.

4-7 Cylindrical waves

- Originated in (ex) a slit.



$$\psi = \frac{A}{\sqrt{\rho}} e^{i(k\rho - \omega t)}$$: $\rho = \sqrt{x^2+y^2}$ if z is the line of symmetry

- does not satisfy the wave equation!

HW#2 Prove it!

Hint : You must derive (or know) $\nabla^2\psi = ?$

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4-8 Electromagnetic waves

· Electric field $\bar{E}$,

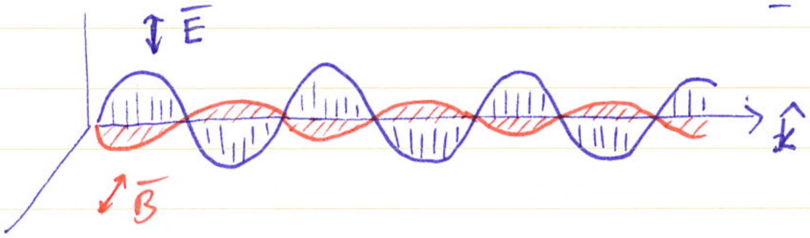
$$\bar{E} = \bar{E}_0 \sin(\bar{k}\cdot\bar{r} - \omega t)$$

· Magnetic field $\bar{B}$,

$$\bar{B} = \bar{B}_0 \sin(\bar{k}\cdot\bar{r} - \omega t)$$

$|E_0| = c\,|B_0|$,

$c = 10^8$ m/s $= \dfrac{1}{\sqrt{\epsilon\mu}}$



· energy density delivered by $\bar{E}$, & $\bar{B}$,

$$u_E = \frac{1}{2}\epsilon_0 E^2 \quad \text{(in free space)}$$

$$u_B = \frac{1}{2}\frac{1}{\mu_0} B^2$$

$$= \frac{1}{2}\frac{1}{\mu_0}\left(\frac{E}{c}\right)^2 = \left(\frac{1}{2}\frac{\epsilon_0\mu_0}{\mu_0}\right)E^2 = u_E$$

∴ total energy density u,

$$u = u_E + u_B = 2u_B = 2u_E = \epsilon_0 E^2 = \left(\frac{1}{\mu_0}\right)B^2$$

· Rate of delivered energy : power, P

$$P = \frac{\text{Energy}}{\Delta t} = \frac{u\,\Delta V}{\Delta t} = \frac{u(Ac)\Delta t}{\Delta t} = ucA$$

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- S : power per unit area S,

$$S = uc$$

$$= \sqrt{u}\sqrt{u} = (\sqrt{\epsilon_0} E)\left(\frac{1}{\sqrt{\mu_0}} B\right)$$

$$= \frac{\epsilon_0}{\sqrt{\epsilon_0 \mu_0}} EB = \epsilon_0 c EB$$

$$\therefore S = \epsilon_0 c^2 EB \quad \text{(poynting vector)}$$

$$\bar{S} = \epsilon_0 c^2 \bar{E} \times \bar{B}$$

. Due to rapid oscillation for VF ($f \sim 10^{15}$ Hz), average power per unit area is considered:

Irradiance, E_e

$$E_e = \langle |\bar{S}| \rangle = \epsilon_0 c^2 \langle E_0 B_0 \sin^2(\bar{k} \cdot \bar{r} - \omega t) \rangle$$

$$= \frac{1}{2} \epsilon_0 c^2 E_0 B_0$$

$$= \frac{1}{2} \epsilon_0 c E_0^2$$

$$= \frac{1}{2} \left(\frac{c}{\mu_0}\right) B_0^2$$